of the potential barrier, which the α-particle has to cross. Outside the potential barrier the wave function is again oscillatory and has a small but finite amplitude. In other words, it means that there is a small but finite probability that the α-particle originally within the nucleus may be found outside the nucleus. This unique quantum phenomenon is known as **Tunnel effect.**

Q. 24.48. Give theoretical basis of α-emission and discuss Gamow's theory of emission of α-particle from a radioactive substance. How far the theoretical results agree with experiments?

(Calicut. U. 2002, 2001; P.U. 1997; Pbi. U. 1999, 1993; G.N.D.U. 1991; H.P.U. 1995)

Ans. Alpha emission. It can be proved that the height of the potential barrier for a U^{238} nucleus is about 26 MeV. According to classical theory, the minimum energy which an α-particle within the U^{238} nucleus, must posses to be able to escape from the nucleus is 26 MeV. But the energies carried by the α-particles emitted by radioactive nuclei are much less than the height of the potential barrier of the respective nuclei. Thus it is very difficult to understand how the α-particle lying within the nucleus can jump over a potential barrier which is more than twice as high as the total energy of the α-particle. However, the escape of an α-particle from a radioactive nucleus can be explained on the basis of quantum theory by the application of wave-mechanics.

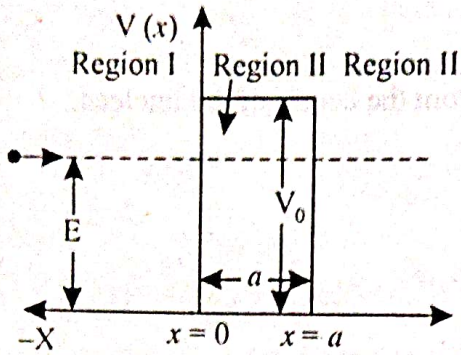

Fig. 24.8

For simplicity, we shall consider a one dimensional coulomb potential barrier of rectangular shape of constant height and width.

Potential barrier of constant height and width. Let a particle of *reduced* mass m be incident on a potential barrier of height V_0 and width a as shown in Fig. 24.8. The energy of the particle is E and $E < V_0$. On both sides of the barrier $V = 0$ which means that no forces act on the particle in regions I and III. The barrier is thus represented by a potential of the form

$V(x) = 0$ for $x < 0$ Region I

$V(x) = V_0$ for $0 \le x \le a$ Region II

$V(x) = 0$ for $x > a$ Region III

According to Quantum mechanics the motion of the particle is governed by Schrodinger's wave equation for regions I, II and III. As the potential V_0 is independent of time, we have to solve Schrodinger's time independent wave equation to analyse the behaviour of the particle.

It is given by

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0 \qquad ...(i)$$

(A) For region I. $(x < 0)$, $V(x) = 0$. Let in this region ψ be denoted by ψ_1.

Substituting $V(x) = 0$ in equation (i), we have

$$\frac{d^2\psi_1}{dx^2} + \frac{2m}{\hbar^2}E\psi_1 = 0$$

or $$\frac{d^2\psi_1}{dx^2} + k_1^2\psi_1 = 0 \qquad ...(ii)$$

where $$k_1^2 = \frac{2mE}{\hbar^2}$$

The general solution of equation (ii) is given by

$$\psi_1 = Ae^{ik_1x} + Be^{-ik_1x}$$

where A and B are arbitrary constants.

The first term in equation (*iii*) represents the incident wave ψ_{in} and the second term the reflected wave ψ_{re}

$$\therefore \quad \psi_{in} = Ae^{i k_1 x} \qquad ...(iv)$$

and $$\psi_{re} = Be^{-i k_1 x} \qquad ...(v)$$

(B) For region II. $(0 \le x \le a)$, $V(x) = V_0$. Let in this region ψ be denoted by ψ_{II}. Substituting the value of $V = V_0$ in equation (*i*), we have

$$\frac{d^2\psi_{II}}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\,\psi_{II} = 0$$

but $E < V_0$ and hence $(E - V_0)$ is a negative quantity. Putting $(E - V_0) = -(V_0 - E)$, we have

$$\frac{d^2\psi_{II}}{dx^2} - \frac{2m}{\hbar^2}(V_0 - E)\,\psi_{II} = 0$$

or $$\frac{d^2\psi_{II}}{dx^2} - k_2^2\,\psi_{II} = 0 \qquad ...(vi)$$

where $$k_2^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

The general solution of equation (*vi*) is given by

$$\psi_{II} = Ce^{-k_2 x} + De^{k_2 x} \qquad ...(viii)$$

where C and D are arbitrary constants.

(C) For region III. $(x > a)$, $V(x) = 0$. Let in this region ψ be denoted by ψ_{III}. Substituting $V = 0$ in equation (*i*), we have

$$\frac{d^2\psi_{III}}{dx^2} + \frac{2mE}{\hbar^2}\psi_{III} = 0$$

or $$\frac{d^2\psi_{III}}{dx^2} + k_1^2\,\psi_{III} = 0 \qquad ...(vii)$$

where $$k_1^2 = \frac{2mE}{\hbar^2}$$ (as for region I)

The general solution of equation (*vii*) is given by

$$\psi_{III} = Fe^{i k_1 x} + Ge^{-i k_1 x} \qquad ...(ix)$$

where F and G are arbitrary constants.

The function $Ge^{-i k_1 x}$ represents a wave moving to the left in $-X$ direction. But there is neither a source of particles in region III nor a further potential step to produce a reflected wave. This means, there is no reflected wave and, therefore, $G = 0$

Substituting $G = 0$ in equation (*ix*), we get

$$\psi_{III} = Fe^{i k_1 x} \qquad ...(x)$$

This is the transmitted wave in region III and may be represented as

$$\psi_{tr} = Fe^{i k_1 x} \qquad ...(xi)$$

(D) Determination of constants. To evaluate the constant A, B, C and D in terms of F, we apply the boundary conditions which are :

1. The wave function ψ must be continuous at the boundary $x = 0$

$$\therefore \quad \psi_I(0) = \psi_{II}(0)$$

$$Ae^{ik_1x_0} + Be^{ik_1x_0} = Ce^{-k_2x_0} + De^{k_2x_0}$$

or $$A + B = C + D \qquad [\because x_0 = 0]$$ ...(i

or

2. The wave function ψ must be continuous at the boundary $x = a$.

$$\psi_{II}(a) = \psi_{III}(a)$$

$$Ce^{-k_2a} + De^{k_2a} = Fe^{ik_1a}$$ ...(ii

or

3. The first order derivatives of the wave function ψ *i.e.*, $\frac{d\psi}{dx}$ must be continuous at $x = 0$.

$$\left|\frac{d\psi_I}{dx}\right|_{x=0} = \left|\frac{d\psi_{II}}{dx}\right|_{x=0}$$

$\therefore$ $$[ik_1Ae^{ik_1x} - ik_1Be^{-ik_1x}]_{x=0} = [-k_2Ce^{-2k_2x} + k_2De^{k_2x}]_{x=0}$$

or $$ik_1(A - B) = -k_2(C - D)$$

or $$A - B = \frac{-k_2}{ik_1}(C - D) = \frac{ik_2}{k_1}(C - D)$$ ...(x

or

4. The first order derivatives of the wave function ψ *i.e.*, $\frac{d\psi}{dx}$ must be continuous at $x = a$

$$\left|\frac{d\psi_{II}}{dx}\right|_{x=a} = \left|\frac{d\psi_{III}}{dx}\right|_{x=a}$$

or $$[-k_2Ce^{-k_2x} + k_2De^{k_2x}]_{x=a} = \left|ik_1Fe^{ik_1x}\right|_{x=a}$$

or $$-k_2Ce^{-k_2a} + k_2De^{k_2a} = ik_1Fe^{ik_1a}$$

or $$-Ce^{-k_2a} + De^{k_2a} = i\frac{k_1}{k_2}Fe^{ik_1a}$$ ...(

Adding equations (*xiii*) and (*xv*), we have

$$2De^{k_2a} = \left(1 + i\frac{k_1}{k_2}\right)Fe^{ik_1a}$$

or $$D = \frac{F}{2}e^{ik_1a}\left(1 + \frac{ik_1}{k_2}\right)e^{-k_2a}$$ ...(x

Subtracting equation (*xv*) from (*xiii*), we get

$$2Ce^{-k_2a} = \left(1 - \frac{ik_1}{k_2}\right)Fe^{ik_1a}$$

or $$C = \frac{F}{2}e^{ik_1a}\left(1 - \frac{ik_1}{k_2}\right)e^{k_2a}$$ ...(x

Again adding equation (*xvi*) and (*xvii*) , we have

$$C + D = \frac{F}{2}e^{ik_1a}\left[(e^{k_2a} + e^{-k_2a}) - i\frac{k_1}{k_2}(e^{k_2a} - e^{-k_2a})\right]$$

$$= Fe^{ik_1a}\left[\left(\frac{e^{k_2a} + e^{-k_2a}}{2}\right) - i\frac{k_1}{k_2}\left(\frac{e^{k_2a} - e^{-k_2a}}{2}\right)\right]$$

$$= Fe^{ik_1a}\left[\cosh(k_2a) - i\frac{k_1}{k_2}\sinh(k_2a)\right]$$ ...(x

Also subtracting equation (*xvi*) from (*xvii*), we have

$$C - D = \frac{F}{2} e^{ik_1 a}\left[(e^{k_2 a} - e^{-k_2 a}) - i\frac{k_1}{k_2}(e^{k_2 a} + e^{-k_2 a})\right]$$

$$= Fe^{ik_1 a}\left[\sinh(k_2 a) - i\frac{k_1}{k_2}\cosh(k_2 a)\right] \quad ...(xix)$$

Substituting the value of $(C + D)$ from equation (*xviii*) in (*xii*), we have

$$A + B = Fe^{ik_1 a}\left[\cosh(k_2 a) - i\frac{k_1}{k_2}\sinh(k_2 a)\right] \quad ...(xx)$$

Also substituting the value of $(C - D)$ from equation (*xix*) in (*xiv*), we have

$$A - B = i\frac{k_2}{k_1}\left[Fe^{ik_1 a}\left\{\sinh(k_2 a) - i\frac{k_1}{k_2}\cosh(k_2 a)\right\}\right]$$

$$= Fe^{ik_1 a}\left[\cosh(k_2 a) + i\frac{k_2}{k_1}\sinh(k_2 a)\right] \quad ...(xxi)$$

Adding (*xx*) and (*xxi*), we get

$$A = Fe^{ik_1 a}\left[\cosh(k_2 a) + \frac{i}{2}\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)\sinh(k_2 a)\right] \quad ...(xxii)$$

Subtracting (*xxi*) from (*xx*)

$$B = -\frac{i}{2} Fe^{ik_1 a}\left[\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)\sinh(k_2 a)\right] \quad ...(xxiii)$$

Transmission Co-efficient. The transmission co-efficient T is defined as

$$T = \frac{\text{Transmitted probability density current}}{\text{Incident probability density current}} = \frac{S_{tr}}{S_{in}}$$

Now $S_{tr} = v_{III}\,\psi_{tr}\,\psi_{tr}^* = v_1\,\psi_{tr}\,\psi_{tr}^* \quad [\because v_{III} = v_1]$

$\therefore$ $S_{tr} = v_1\,Fe^{ik_1 x}\,F^* e^{-ik_1 x} = v_1\,FF^*$

and $S_{in} = v_1\,A_e^{ik_1 x}\,A^* e^{-ik_1 x} = v_1\,AA^*$

$\therefore$ $$T = \frac{FF^*}{AA^*} = \frac{1}{\cosh^2(k_2 a) + \frac{1}{4}\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)^2 \sinh^2(k_2 a)}$$

$$= \frac{1}{1 + \sinh^2(k_2 a) + \left(\frac{k_2^2 - k_1^2}{2k_1 k_2}\right)^2 \sinh^2 k_2 a}$$

$$= \frac{1}{1 + \sinh^2(k_2 a)\left[1 + \left(\frac{k_2^2 - k_1^2}{2k_1 k_2}\right)^2\right]}$$

$$= \frac{1}{1 + \left(\frac{k_2^2 + k_1^2}{2k_1 k_2}\right)^2 \sinh^2(k_2 a)} \quad ...(xxiv)$$

We know that $\sinh(k_2 a) = \frac{e^{k_2 a} - e^{-k_2 a}}{2} = \frac{e^{-k_2 a}}{2}[e^{2k_2 a} - 1]$

When $k_2 a >> 1$, 1 can be neglected as compared to $e^{2k_2 a}$

and then $\sinh(k_2 a) = \frac{e^{-k_2 a} \cdot e^{2k_2 a}}{2} = \frac{e^{k_2 a}}{2}$

and $\sinh^2(k_2 a) = \frac{e^{2k_2 a}}{4}$

Substituting this value of $\sinh^2(k_2 a)$ in the expression for T and neglecting 1 as compared to $\sinh^2(k_2 a)$ term, we get

$$T = \frac{1}{\left(\frac{k_2^2 + k_1^2}{2k_1 k_2}\right)^2 \frac{e^{2k_2 a}}{4}} = 4\left(\frac{2k_1 k_2}{k_2^2 + k_{1_2}}\right)^2 e^{-2k_2 a}$$

$$= \left(\frac{4k_1 k_2}{k_2^2 + k_1^2}\right)^2 e^{-2k_2 a} \quad \ldots(xxv)$$

The variation in the co-efficient of the exponential term of Eq. (*xxv*) with E and V_0 is negligible compared with the variation in the exponential term itself. Further the co-efficient is never far from unity. Therefore to a very fair approximation, the transmission probability

$$T = e^{-2k_2 a} \quad \ldots(xxvi)$$

Gamow's theory of α-decay. This theory was developed in 1928 by Gamow in collaboration with Gurney and Condon. According to this theory.

1. An α-particle may exist as an entity within the heavy nucleus.

2. The α-particle is in constant motion and is contained in the nucleus by the surrounding potential barrier. It bounces back and forth from the barrier walls. In each collision with the '*wall*' there is a definite probability given by equation (*xxvi*) that the particle will leak through the barrier. In fact, the α-particle within the nucleus must present itself again and again at the barrier surface until conditions are ripe for penetration or leakage. In $_{92}U^{238}$ the α-particle must make 10^{38} tries *i.e.*, 10^{22} tries per second for 10^{16} seconds or 10^9 years before it escapes.

Let ν be the frequency with which the α-particle collides with the walls in order to escape from the nucleus and T the transmission probability in each collision, then

Radioactive constant λ = decay probability per unit time

$$= \nu T$$

Suppose at any time only one α-particle exists as such in the nucleus and that it moves back and forth along the nuclear diameter then

$$\nu = \frac{v}{2r_0}$$

where v is the velocity of the α-particle when it eventually leaves the nucleus and r_0 is the nuclear radius.

Hence $$\lambda = \frac{v}{2r_0} T \quad \ldots(xxvii)$$

The value of transmission probability for a barrier of constant height V_0 and width a as given by equation (*xxvi*) is

$$T = e^{-2k_2 a}$$

where $k_2 = \frac{\sqrt{2m\,(V_0 - E)}}{h}$

But an α-particle within the nucleus has to overcome a *barrier of varying height* as shown in Fig. 24.6.

As the potential is not constant in the region $x = r_0$ to $x = r_1$, we can divide it in a series of small steps each of thickness dx, then the probability of transmission through a small thickness dx is given by

$$T = e^{-2k_2(x)\,dx}$$

The total probability is the product of individual probabilities of all the small steps from $x = r_0$ to $x = r_1$

Hence $$T = e^{-2\int_{r_0}^{r_1} k_2(x)\,dx}$$

or $$\ln T = -2\int_{r_0}^{r_1} k_2(x)\,dx$$

r_1 is the distance from the centre of the nucleus where $V_x = E$. The kinetic energy E is greater than the potential energy V_x for values of $x > r_1$. So, if the α-particle can get past r_1, it will permanently escape from the nucleus.

Now $V_x = \frac{2Ze^2}{4\pi\varepsilon_0 x}$ where Z is the atomic number of the daughter nucleus.

$$\therefore \quad k_2(x) = \frac{\sqrt{2m\,(V_x - E)}}{h} = \left(\frac{2m}{h^2}\right)^{1/2}\left[\frac{2Ze^2}{4\pi\varepsilon_0 x} - E\right]^{1/2} \qquad ...(xxviii)$$

As $V_x = E$ when $x = r_1$

$$E = \frac{2Ze^2}{4\pi\varepsilon_0 r_1} \qquad ...(xxix)$$

$$\therefore \quad \frac{2Ze^2}{4\pi\varepsilon_0 x} = \frac{2Ze^2}{4\pi\varepsilon_0 r_1}\frac{r_1}{x} = E\frac{r_1}{x}$$

Substituting in equation (*xxviii*), we get

$$k_2(x) = \left(\frac{2mE}{h^2}\right)^{1/2}\left(\frac{r_1}{x} - 1\right)^{1/2}$$

and $$\ln T = -2\int_{r_0}^{r_1} k_2(x)\,dx = -2\left(\frac{2mE}{h^2}\right)^{1/2}\int_{r_0}^{r_1}\left(\frac{r_1}{x} - 1\right)^{1/2} dx \qquad ...(xxx)$$

To evaluate the integral put $\frac{x}{r_1} = \cos^2\phi$

or $$x = r_1\cos^2\phi$$

$$\therefore \quad dx = -2r_1\cos\phi\sin\phi\,d\phi$$

Hence $$\left(\frac{r_1}{x} - 1\right)^{1/2} dx = -\left(\frac{1}{\cos^2\phi} - 1\right)^{1/2} 2r_1\cos\phi\sin\phi\,d\phi$$

$$= -2r_1\sin^2\phi\,d\phi = -r_1(1 - \cos 2\phi)\,d\phi$$

$$\therefore \quad \int_{r_0}^{r_1}\left(\frac{r_1}{x} - 1\right)^{1/2} dx = -r_1\int_{r_0}^{r_1}(1 - \cos 2\phi)\,d\phi$$

$$= -r_1\left[\phi - \frac{\sin 2\phi}{2}\right]_{r_0}^{r_1} = -r_1\left[\phi - \sin\phi\cos\phi\right]_{r_0}^{r_1}$$

$$= r_1\left[\cos^{-1}\left(\frac{r_0}{r_1}\right)^{1/2} - \left(\frac{r_0}{r_1}\right)^{1/2}\left(1-\frac{r_0}{r_1}\right)^{1/2}\right]$$

$$\therefore \quad \ln T = -2\left(\frac{2mE}{\hbar^2}\right)^{1/2} r_1\left[\cos^{-1}\left(\frac{r_0}{r_1}\right)^{1/2} - \left(\frac{r_0}{r_1}\right)^{1/2}\left(1-\frac{r_0}{r_1}\right)^{1/2}\right]$$

since the potential barrier is relatively wide $r_1 >> r_0$

$$\therefore \quad \cos^{-1}\left(\frac{r_0}{r_1}\right)^{1/2} = \frac{\pi}{2} - \left(\frac{r_0}{r_1}\right)^{1/2}$$

and $$\left(1-\frac{r_0}{r_1}\right)^{1/2} = 1$$

Substituting in equation (*xxxi*), we get

$$\ln T = -2\left(\frac{2mE}{\hbar^2}\right)^{1/2} r_1\left[\frac{\pi}{2} - 2\left(\frac{r_0}{r_1}\right)^{1/2}\right]$$

Hence $$\ln T = -2\left(\frac{2mE}{\hbar^2}\right)^{1/2}\left[r_1\frac{\pi}{2} - 2(r_0 r_1)^{1/2}\right]$$

$\therefore$ From equation (*xxix*) $$r_1 = \frac{2Ze^2}{4\pi\varepsilon_0 E}$$

$$\therefore \quad \ln T = -2\left(\frac{2mE}{\hbar^2}\right)^{1/2}\left[\frac{Ze^2}{4\varepsilon_0 E} - 2\left(\frac{r_0 Ze^2}{2\pi\varepsilon_0 E}\right)^{1/2}\right]$$

$$= \frac{4e}{\hbar}\left(\frac{m}{\pi\varepsilon_0}\right)^{1/2} Z^{1/2} r_0^{1/2} - \frac{e^2}{\hbar\varepsilon_0}\left(\frac{m}{2}\right)^{1/2} ZE^{-1/2}$$

Now $$\frac{4e}{\hbar}\left(\frac{m}{\pi\varepsilon_0}\right)^{1/2} = 2.97 \text{ and } \frac{e^2}{\hbar\,\varepsilon_0}\left(\frac{m}{2}\right)^{1/2} = 3.95$$

$\therefore$ $$\ln T = 2.97\, Z^{1/2} r_0^{1/2} - 3.95\, ZE^{-1/2}$$

where r_0 is in fermi (10^{-15} m) and E is in MeV. The radioactive constant ...(xxx

$$\lambda = \frac{v}{2r_0} T$$

$$\therefore \quad \ln \lambda = \ln\left(\frac{v}{2r_0}\right) + \ln T$$

$$= In\left(\frac{v}{2r_0}\right) + 2.97\, Z^{1/2}\, r_0^{1/2} - 3.95\, ZE^{-1/2}$$

or $$\log_{10}\lambda = \log_{10}\left(\frac{v}{2r_0}\right) + 1.29\, Z^{1/2}\, r_0^{1/2} - 1.72\, ZE^{-1/2} \quad ...(xxx$$

Verification of results. (*i*) A graph between $\log_{10}\lambda$ and $ZE^{-1/2}$ for various α-active nuclei shown in Fig. 24.9. The slope of straight line = – 1.72 as calculated theoretically.

(*ii*) The intercept on the Y-axis gives the value of $\log_{10}\left(\frac{\nu}{2r_0}\right) + 1.29\, Z_{1/2}\, r_0^{1/2}$.

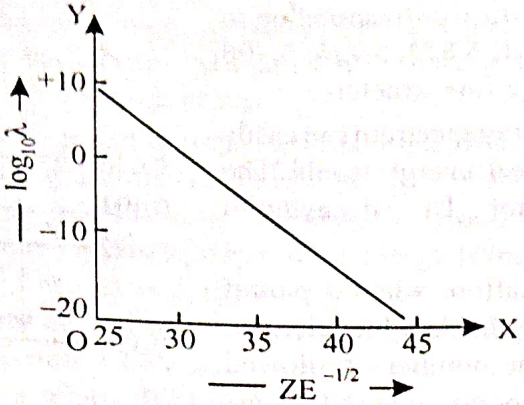

Fig. 24.9

The value of r_0 the nuclear radius calculated from this relation is of the same order as obtained from other experiments like scattering.

(*iii*) Relation (*xxxiii*) verifies *Geiger-Nuttal law.*

(*iv*) This relation also explains why nuclei emitting most energetic α-particles are short lived and those emitting least energetic α-particles are long lived.

Q. 24.49. What is tunnel effect ? How is it applied in α-decay ? (*Calicut. U.* 2003)

Ans. Tunnel effect. The transmission probability that an α-particle may penetrate through the potential barrier is given by

$$T = e^{-2k_2 a} \qquad \text{...(}i\text{) [For proof See Q 24.48. Eq (}xxvi\text{)]}$$

where $k_2^2 = \frac{2m}{\hbar^2}(V_0 - E)$; V_0 = constant height and a the width of the potential barrier, m the reduced mass of the α-particle and E its energy ($E < V_0$).

It is clear from the relation $T = e^{-2k_2 a}$ that although the energy of the α-particle is less than the barrier height and classically it is not possible for the α-particle to cross the barrier height $V_0 > E$, quantum mechanically it is possible as the transmission probability is not zero.

According to Quantum mechanics, α-particle exists as an entity within the heavy nucleus. The α-particle is in constant motion and is contained in the nucleus by the surrounding potential barrier. It bounces back and forth from the barrier walls. In each collision with the '*wall*', there is a *finite probability* given by Eq. (*i*) that the particle will leak through the barrier. In fact the α-particle within the nucleus must present itself again and again at the barrier surface until conditions are ripe for penetration or leakage. In $_{92}U^{238}$ the α-particle must make 10^{38} tries *i.e.* 10^{22} tries per second for 10^{16} seconds or 10^9 years before it escapes.

As there is a finite probability for the α-particle to escape through the potential barrier of height V_0 greater than its energy E, the phenomenon is known as barrier penetration or *tunnel effect.*

Q. 24.50. (*a*) What is meant by fine structure in α-ray spectrum ? How does it lead to the existence of energy levels in nuclei ?

(*b*) What are long range α-particles ?

Ans. (*a*) Fine structure. Accurate energy measurements of the α-particle emitted by radioactive substances indicate that a number of α-emitters show a group of discrete α-particle energies rather than a single energy. This is known as the *fine structure in α-ray energy spectrum.*